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On Optimized Prestressed Trusses

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Introduction

AN earlier publication[1] has presented a basic formulation of the optimization problem for indeterminate structures that are prestressed by component mismatch as well as subjected to multiple independent-load conditions. Illustrative examples, consisting of a stress-limited 3-bar truss[1] and two-span beam,[2] have demonstrated that designs that include optimal selection of mismatch parameters can be significantly lower in weight than their nonprestressed counterparts.

It is the purpose of this Note to extend the examination of the effects of such prestressing by considering several trusses of varying size and complexity, including stress, displacement, and minimum area limits. In particular, attention will be focused on a 10-bar truss, a 25-bar truss, and a 72-bar truss, examples that are encountered most frequently in current optimization studies.[3-8] Redesign of these trusses, incorporating component mismatch, should provide some definitive insight into actual and potential consequences of this prestressing mechanism.

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Synthesis

A truss optimization problem in which element mismatches are included as design parameters may be stated in the basic form[1]: find the components of a vector of design variables

$$\{\tilde{D}\}^T \equiv \{D^T \,|\, D_p^T\} = \{d_1 \ldots d_n \ldots d_N \,|\, d_{N+1} \ldots d_m \ldots d_M\}$$

such that

$$g_k(\tilde{D}) \geq 0 \qquad k = 1,2,\ldots,K \tag{1}$$

and such that weight $W(D)$ is minimized, i.e.,

$$W(D) \rightarrow \min \tag{2}$$

where D is a vector of basic variables (i.e., cross-sectional areas) and D_p is a vector of initial element elongations (changes in length from nominal). Since variables may be linked, integers N and $M-N$ represent the number of independent areas and independent elongations, respectively, where N and $M-N$ are less than or equal to I, I being the total number of truss components.

The inequalities in Eq. (1) represent behavioral and side constraints, which specify that 1) the initial prestress in element i of the unloaded structure, σ_{i0}, $i=1,\ldots,I$, must be bounded by allowable tensile and compressive limits; 2) the stress in element i of the prestressed structure with load condition ℓ superposed, $\sigma_{i\ell}$, $\ell=1,\ldots L$, must be bounded by the allowable tensile and compressive limits; 3) the magnitude of the jth system displacement coordinate in load condition ℓ, $u_{j\ell}$, $j=1,\ldots,J$, $\ell=0, 1,\ldots,L$, must be bounded by allowable limiting values; and 4) the cross-sectional areas must be greater than or equal to minimum allowable values. It may be noted that truss weight [see Eq. (2)], is a function of D only, whereas all constraints except those involving minimum area are functions of $\tilde{D}$.

For the examples considered herein, the design problem given by relations (1) and (2) was solved by use of CONMIN,[9] a computer program based on Zoutendijk's feasible direction algorithm.[10] This algorithm requires gradients of the objective function and constraints, and the CONMIN program has the capability of computing these gradients by finite-differences or of accepting, as part of user-supplied structural analysis subroutines, analytic expressions for these gradients.

Since W is a linear function of d_n, $n=1,\ldots,N$, an analytic expression for the gradient of $W(D)$ is easily obtained. The use of analytic constraint gradients is also desirable in order to avoid the excessive number of structural analyses that would be required for their finite-difference computation. (Note that the total number of independent variables in this class of problem can be relatively large; e.g., $2N$ for linking based only on structural symmetry.) The development of analytic expressions for the components of constraint gradients (i.e., derivatives of stresses and displacements with respect to design variables), for the prestressed structures under consideration, is shown in the Appendix.

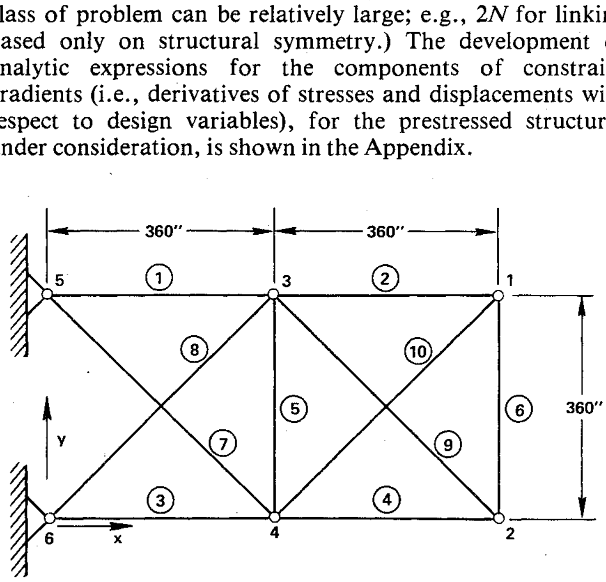

Fig. 1 Planar 10-bar cantilever truss.

Table 1 Design results and comparisons

	Weight (lb.)					
	10-bar truss		25-bar truss		72-bar truss	
	Stress, min. area	Stress, displ., min. area	Stress, min. area	Stress, displ., min. area	Stress, min. area	Stress, displ., min. area
Nonprestressed	1593.2[a]	5107.3[b]	344.0[7]	548.5[8]	96.6[7]	379.8[8]
Prestressed	1597.6	3533.8	272.8	545.6	94.2	345.2
% Difference	–	30.8	20.7	–	2.5	9.1

[a] Problem 1A, Ref. 8. [b] Problem 3, Ref. 8.

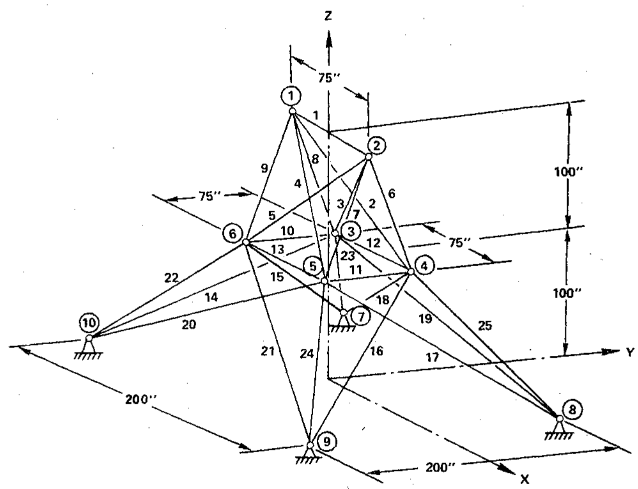

Fig. 2 Twenty-five-bar space truss.

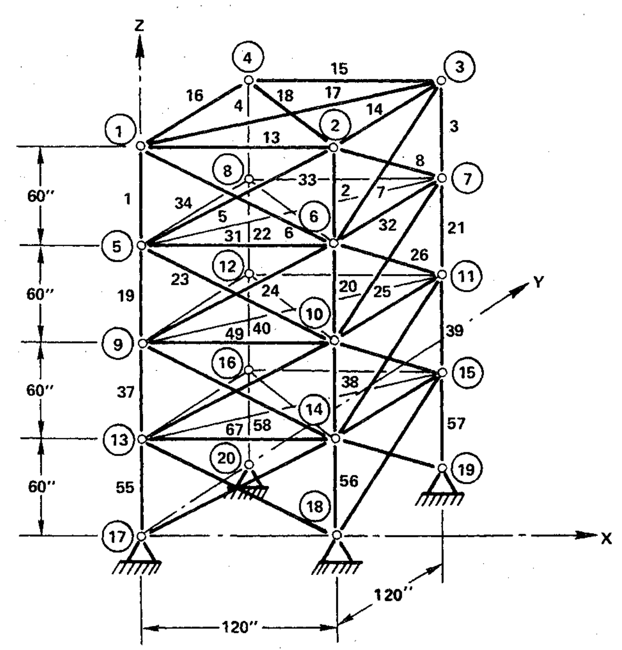

Fig. 3 Seventy-two bar space truss. (Note: for the sake of clarity not all elements are drawn in this figure.)

Examples

Figures 1-3 illustrate the basic forms of the specific truss design examples utilized in this study. Details of configuration, loadings, allowable stresses and displacements, and minimum area limits are contained in the references cited.

Table 1 compares minimum weights previously obtained for the non-prestressed trusses with those of the current study,

Table 2 Design variables for optimized prestressed trusses.

Example	Independent Element No.	Area Variables, in.^{2}		Prestress Variables, in.	
		Stress, Min. Area	Stress, Displ., Min. Area	Stress, Min. Area	Stress, Displ., Min. Area
10-Bar Truss	1	7.8834	21.3100	0.0956	-0.3813
	2	0.1000	1.1028	0.0183	-0.2900
	3	8.1771	22.0780	0.0968	0.6340
	4	3.9391	3.3220	0.0185	0.0485
	5	0.1000	0.1000	0.1145	0.0057
	6	0.1000	1.0960	0.0184	-0.3326
	7	5.8738	13.5480	-0.1366	-0.4180
	8	5.4705	12.1940	-0.1357	0.0606
	9	5.5781	7.4588	-0.0261	-0.0686
	10	0.1045	1.5553	-0.0258	0.4100
25-Bar Truss	1	0.2512	0.0100	0.0509	-0.0006
	2	0.5086	2.0463	-0.1772	0.0020
	3	1.5700	3.0159	0.1450	-0.0017
	4	0.0910	0.0100	0.1158	0.0284
	5	0.5281	0.0100	0.1401	0.0300
	6	0.1341	0.6900	-0.2098	-0.0512
	7	0.4693	1.6281	-0.1615	-0.0557
	8	2.3276	2.6532	-0.2736	0.0784
72-Bar Truss	1	0.1906	0.1000	-0.0009	-0.0378
	2	0.1000	0.4795	-0.0025	0.1155
	3	0.1000	0.2649	0.0085	0.0082
	4	0.1000	0.3618	-0.0038	0.0363
	5	0.1541	0.5709	-0.0702	0.0333
	6	0.1000	0.4793	0.1571	-0.0433
	7	0.1000	0.1000	-0.0433	-0.0097
	8	0.1000	0.1000	-0.0218	-0.0160
	9	0.1627	1.2478	-0.0750	0.0062
	10	0.1000	0.4781	0.1678	0.0170
	11	0.1000	0.1000	-0.0840	0.0051
	12	0.1000	0.1000	-0.0433	-0.0047
	13	0.2630	1.8444	-0.0575	0.0168
	14	0.1000	0.4758	0.1288	-0.0063
	15	0.1000	0.1000	-0.0784	-0.0022
	16	0.1000	0.1000	-0.0384	-0.0019

and Table 2 presents the corresponding optimum values o design variables for the prestressed trusses. The elemen numbering system in Table 2 is consistent with that used i Ref. 8. It should be noted that the data in Table 1, which i taken from Ref. 8, are also based on the use of the CONMIN program; additional results in Ref. 8 based on an alternat optimization algorithm (NEWSUMT) show weights slightl lower than those listed here.

Conclusions

A study of the results in Tables 1 and 2 indicates tl following:

1) Stress-limited designs that are not fully stressed and d not have many active minimum-area constraints (25-bar trus can be significantly improved by prestressing. Conversel designs that are nearly fully stressed and/or have many acti minimum-area constraints (10-bar and 72-bar trusses) cann be significantly improved by prestressing.

2) Displacement-limited designs can be improved l prestressing only if it is possible for component mismatches produce an initial deformation pattern opposite to t deformation patterns caused by loads; thus, unsymmetrica loaded structures (10-bar truss), or unsymmetrically limit structures, may be improved by prestressing, whereas mc symmetrically loaded structures (25-bar truss) may be l affected by prestressing (the latter condition may be diffic to identify in practice).

3) Initial deformations of relatively small magnitude can have significant effects on stresses and displacements of loaded structures. Consequently, unspecified or inadvertent initial deformations may reduce the reliability of structures optimized using procedures that do not adequately account for the effects of such deformations.

Appendix: Derivatives of Stresses and Displacements for Prestressed Trusses

The stiffness relations between applied forces and developed displacements for a linear elastic structure are of the form

$$[K]\ \{u\} = \{F\} = \{F^m\} + \{F^0\} \tag{A1}$$

where $[K]$ is the stiffness matrix, $\{F\}$ the vector of equivalent nodal loads and $\{u\}$, the vector of nodal displacements. The forces are the sum of the applied mechanical loads $\{F^m\}$ and the initial prestress loads $\{F^0\}$, where expressions for $\{F^0\}$ are given in numerous references.[11,12]

Differentiating Eq. (A1) with respect to design variable d_j gives, for load condition ℓ,

$$[K]\frac{\partial\{u\}_\ell}{\partial d_j} + \frac{\partial[k]}{\partial d_j}\{u\}_\ell = \frac{\partial\{F^m\}_\ell}{\partial d_j} + \frac{\partial\{F^0\}}{\partial d_j} \tag{A2}$$

The mechanical loads are independent of both the area design variables and the prestress design variables; consequently, $\partial\{F^m\}_\ell/\partial d_j = 0$.

The element areas A_i are linked to the area design variables d_n through a linking matrix with components a_{in}

$$A_i = a_{in} d_n \quad n = 1,2,\ldots,N \quad i = 1,2,\ldots,I \tag{A3}$$

The gradient of the prestress forces with respect to area design variables is, for truss elements

$$\frac{\partial\{F^0\}}{\partial d_n} = \sum_{i\epsilon n} a_{in}\frac{\partial\{F^0\}}{\partial A_i} = \sum_{i\epsilon n} a_{in} E u_i^0 \{B\}_i \tag{A4}$$

where E is the elastic modulus, u_i^0 the initial displacement, and $\{B\}_i$ the linear strain-displacement vector.

The gradient of the displacement with respect to the area design variables is obtained by substituting Eq. (A4) into Eq. (A2) and solving for $\partial\{u\}_\ell/\partial d_n$; this gives

$$\frac{\partial\{u\}_\ell}{\partial d_n} = [K]^{-1}\left(\sum_{i\epsilon n} a_{in} E u_i^0 \{B\}_i - \frac{\partial[K]}{\partial d_n}\{u\}_\ell\right) \tag{A5}$$

where the stiffness gradients $\partial[K]/\partial d_n$ are constant and need to be calculated only once.

The member initial displacements u_i^0 are assumed to be linked to the prestress design variables d_m through a linking matrix with components $\hat{a}_{i,m-N}$,

$$u_i^0 = \hat{a}_{i,m-N} d_m \qquad m = N+1,\ldots,M \tag{A6}$$

where M and N are integers defined previously.

The gradient of the prestress force with respect to the prestress design variables is

$$\frac{\partial\{F^0\}}{\partial d_m} = \sum_{i\epsilon m} \hat{a}_{i,m-N}\frac{\partial\{F^0\}}{\partial u_i^0} = \sum_{i\epsilon m} \hat{a}_{i,m-N} E A_i \{B\}_i \tag{A7}$$

The gradient of the displacements with respect to the prestress design variables is, from Eqs. (A2) and (A7)

$$\frac{\partial\{u\}_\ell}{\partial d_m} = [K]^{-1}\sum_{i\epsilon m} \hat{a}_{i,m-N} E A_i \{B\}_i \tag{A8}$$

where the gradient of the stiffness matrix with respect to the prestress variables has been taken as identically zero.

The stress in member i under load condition ℓ is

$$\sigma_{i\ell} = E(\epsilon_{i\ell} - \epsilon_{i\ell}^0) = E\left(\{B\}_i^T\{u\}_\ell - \frac{u_i^0}{L_i}\right) \tag{A9}$$

where L_i is the length of the member.

The gradient of the stress with respect to the area design variables is

$$\frac{\partial\sigma_{i\ell}}{\partial d_n} = E\left(\{B\}_i^T\frac{\partial\{u\}_\ell}{\partial d_n} - \frac{1}{L_i}\frac{\partial u_i^0}{\partial d_n}\right) = E\{B\}_i^T\frac{\partial\{u\}_\ell}{\partial d_n} \tag{A10}$$

where the displacement gradient $\partial\{u\}_\ell/\partial d_n$ is given by Eq. (A5). The gradient of the stress with respect to the prestress design variables is

$$\frac{\partial\sigma_{i\ell}}{\partial d_m} = E\left(\{B\}_i^T\frac{\partial\{u_\ell\}}{\partial d_m} - \frac{1}{L_i}\hat{a}_{i,m-N}\right) \tag{A11}$$

where the displacement gradient $\partial\{u\}_\ell/\partial d_m$ is given by Eq. (A8).

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Eigenrelations in Structural Dynamics

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Introduction

THE second-order system of ordinary differential equations representing free vibrations for a system with n degrees of freedom

$$[M]\ \{\ddot{x}\} + [C]\ \{\dot{x}\} + [K]\ \{x\} = \{0\} \tag{1}$$

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